# Interlacing Families and Bipartite Ramanujan Graphs

Adam W. Marcus

Crisply, LLC
Yale University
adam.marcus@yale.edu

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Joint work with:

Dan Spielman
Yale University

Nikhil Srivastava
Microsoft Research, India

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# Goals

In this talk I plan to

1. Give a brief introduction to graph expansion
2. Give a brief survey of what is known about Ramanujan families
3. Motivate the approach we took in trying to find Ramanujan families
4. Introduce a technique for showing the existence of combinatorial objects we call "the method of interlacing polynomials"
5. Use this to prove the existence of Ramanujan families of arbitrary degree
6. Discuss some related open questions

not necessarily (but mostly) in this order.

# Simplifications

Throughout the talk, the following things will hold:

- $G = (V, E)$ will be a $d$-regular graph
- We will assume a fixed ordering on $E = \{e_1, \ldots, e_m\}$.
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And *please* interrupt if there are any questions.

# Outline

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Used throughout computer science.

- Error-correcting codes
- Pseudorandom generators
- Computational complexity
  - PCP theorem (Dinur 2007)
  - SL=L (Reingold 2005)

# Types of Expansion

There are different definitions of "expansion"

- Edge expansion

$$h_E(G) = \min_{0<|S|\leq n/2} \frac{|\{e \in E : |e \cap S| = 1\}|}{|S|}$$

- Vertex expansion

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For constant degree graphs, these are interchangable (up to a constant).

# Adjacency Matrix

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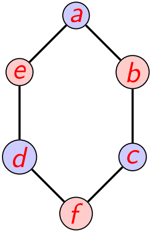


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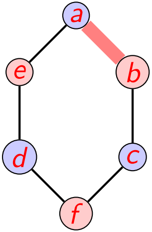


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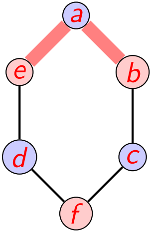


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1. $A_{i,j} = 1$ if and only if $\{v_i, v_j\} \in E$
2. Since the graph is $d$-regular, each row sums to $d$

# The spectrum

Since $A$ is symmetric, it has all real eigenvalues

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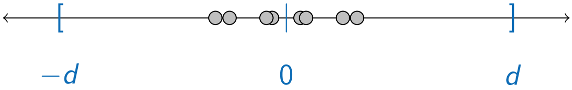

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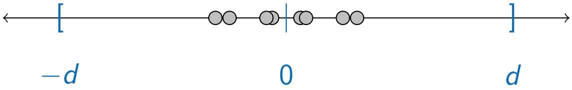


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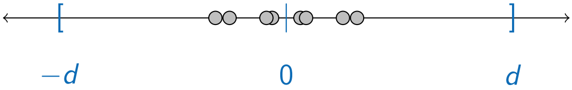


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But in practice, we need *big* graphs with *small* degree.

In particular, we would like *infinite families* of such graphs.

# Outline

# What can we hope for?

Theorem (Alon–Boppana (1986))

*For every* $\epsilon > 0$*, there exists an* $N$ *such that any* $d$*-regular graph on* $N$ *vertices has a non-trivial eigenvalue* $\alpha$ *with* $|\alpha| \geq 2\sqrt{d-1} - \epsilon$

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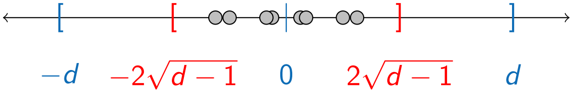
A $d$-regular graph that has all non-trivial eigenvalues inside the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ is called *Ramanujan*.

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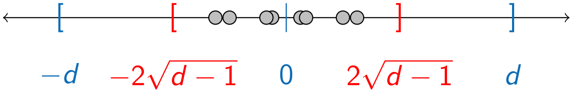


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We will call an infinite collection of $d$-regular Ramanujan graphs a *Ramanujan family*.

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The largest eigenvalue of the $d$-regular infinite tree is $2\sqrt{d-1}$ (the smallest is $-2\sqrt{d-1}$).

So being Ramanujan can be seen as being a good (finite) approximation of a $d$-regular infinite tree.

# Previous results

Theorem (Margulis, Lubotzky–Phillips–Sarnak (1988))

*Ramanujan families exist for $d = p + 1$ where $p$ is a prime number.*

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Answer: Almost no.

Theorem (Friedman (2008))

*A randomly chosen* $d$*-regular graph has its non-trivial eigenvalues in the interval*

$$[-2\sqrt{d-1} - \epsilon, 2\sqrt{d-1} + \epsilon]$$

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So are Ramanujan families "special" or are they just "everywhere"?

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Two caveats:

1. The families guaranteed in our proof will be families of *bipartite* graphs
2. Despite showing that such families are "everywhere", we are not actually able to construct one

Instead, we use a new technique for showing existence of combinatorial objects we call "the method of interlacing polynomials".

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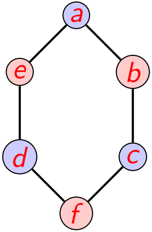
4. Such families were known to exist but only for certain values of $d$ and only for algebraic reasons
5. We will prove the existence of (bipartite) Ramanujan families of every degree

# Outline

# General Idea

Start with a good graph.



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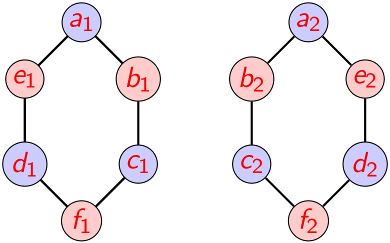


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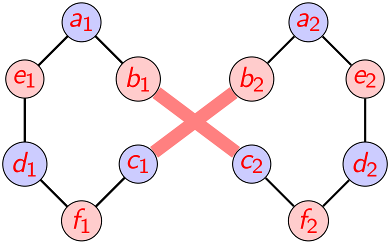


And make a copy of it. And perturb it.

Want to find perturbations that cause new graph to be good.

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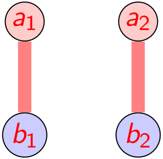


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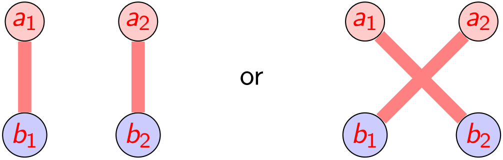


Positive Edge Lift    Negative Edge Lift

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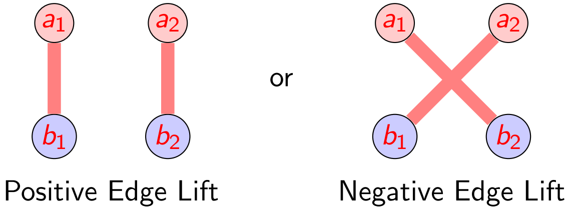
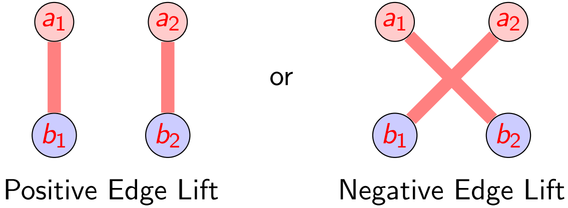



for a total of $2^{|E|}$ possible 2-lifts.

We will refer to a 2-lift by its *signing* $s \in \{\pm\}^m$ and refer to the corresponding (lifted) graph as $G_s$.

# Important properties

Let $e_i = (a, b)$ be an edge in $G$. Then (depending on the value of $s_i$) we have either

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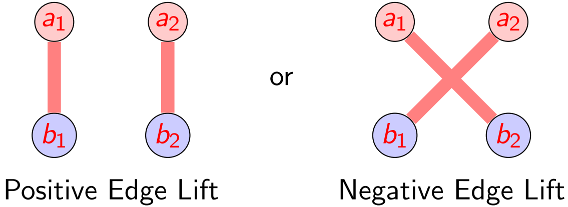


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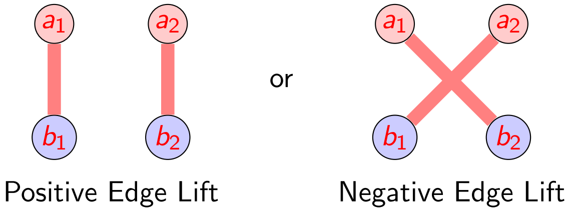


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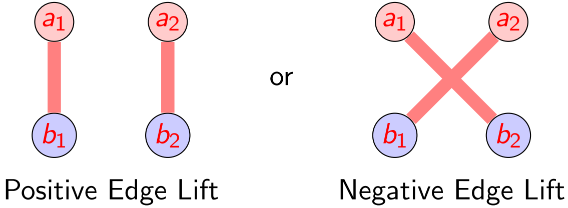


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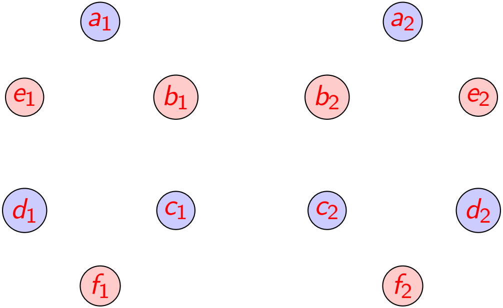
So, in particular, if $G$ is bipartite and $d$-regular, then $G_s$ is bipartite and $d$-regular (for all signings $s$).

# Examples

For example:



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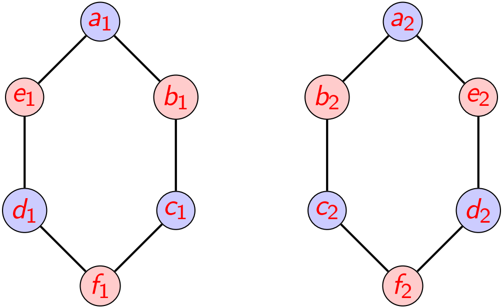


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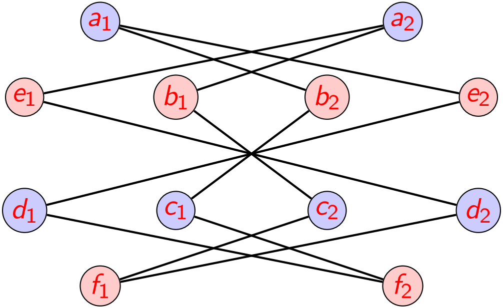


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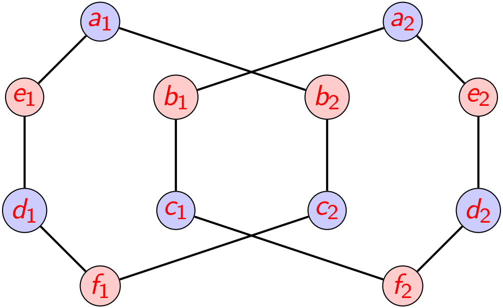


1. All positive
2. All negative
3. $(a, b)$ and $(c, f)$ negative (rest positive)

# Signed Adjacency Matrix

Let $A$ be the adjacency matrix of $G$ and $s$ a signing of $G$.

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Multiply each value in $A$ by the corresponding sign from $s$.

This is called the *signed adjacency matrix* $A_s$.

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# Main Eigenvalue lemma

## Theorem (Bilu–Linial (2006))

*Let $G$ be* **any** *graph, $s$ a signing of $G$ and $G_s$ the 2-lift of $G$ corresponding to $s$.*

*Then the eigenvalues of $A(G_s)$ (the new adjacency matrix) are the union of the eigenvalues of $A$ (the original adjacency matrix) and the eigenvalues of $A_s$ (the signed adjacency matrix).*

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Therefore if $G$ was Ramanujan and the eigenvalues of $A_s$ were in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$, then $G_s$ would be Ramanujan.

## Conjecture (Bilu–Linial (2006))

***Every** $d$-regular graph contains a signing $s$ for which all of the eigenvalues of $A_s$ (the signed adjacency matrix) lie inside the interval*

$$[-2\sqrt{d-1}, 2\sqrt{d-1}]$$

# Implications

If the conjecture was true, this would imply the existence of Ramanujan families of degree $d$ (for any $d$).

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1. Start with a $d$-regular Ramanujan graph $G$
2. Find a signing $s$ for which all eigenvalues of $A_s$ lie in the range $[-2\sqrt{d-1}, 2\sqrt{d-1}]$
3. Perform the 2-lift associated with the signing to get graph $G_s$ (with twice as many vertices).
4. Then $G_s$ is Ramanujan, so iterate

Note: we can always start with $G = K_{d+1}$ or $G = K_{d,d}$.

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Note: we can always start with $G = K_{d+1}$ or $G = K_{d,d}$.

We will prove the conjecture for every *bipartite* graph $G$.

Since 2-lifts preserve bipartiteness, the same proof applies.

# Bipartite Adjacency Matrices

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$$\left(\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right)$$

causing eigenvalues/vectors to come in pairs

$$v_i = [u_i \mid u_i] \quad \text{and} \quad v_{n-i} = [u_i \mid -u_i]$$

for $1 \leq i \leq n/2$ and so the eigenvalues satisfy $\lambda_i = -\lambda_{n-i}$.

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## Corollary

*A bipartite graph $G$ has all of its non-trivial eigenvalues in the range*

$$[-2\sqrt{d-1}, 2\sqrt{d-1}]$$

*if and only if it has all non-trivial eigenvalues at most $2\sqrt{d-1}$.*

# Intermission

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## Theorem

*Every $d$-regular bipartite graph has a signing $s$ such that the largest eigenvalue of the signed adjacency matrix is at most $2\sqrt{d-1}$.*

Side note: a random signing does not (in general) work:

$$\mathbb{E}_s\left[\|A_s\|\right] \gg 2\sqrt{d-1}$$

as noted by Bilu and Linial.

# Step By Step

Our approach will be to build the signing one edge at a time and see what happens to the eigenvalues.

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To "keep track" we will use a generalization of the *characteristic polynomial* of a matrix

$$\chi_M(x) = \det(xI - M) = \prod_i (x - \lambda_i)$$

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where the $\lambda_i$ are the eigenvalues of $M$.

Given a vector $t \in \{\pm\}^k$ for $k \leq m$, define the *partial assignment polynomial*

$$p_t(x) := \mathbb{E}_{s \in \{\pm\}^m} \left[ \chi_{A_s}(x) \mid s_1 = t_1, \ldots, s_k = t_k \right]$$

# Some notes

For $s \in \{\pm\}^m$, the partial assignment polynomial is just the characteristic polynomial of the matrix $A_s$

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is the expected characteristic polynomial over all possible signed adjacency matrices.

Miraculously, the expected characteristic polynomial is something we can get our hands on.

# Miracle 1

Theorem (Godsil–Gutman (1981))

*For any graph $G$,*

$$\mathbb{E}_{s\in\{\pm\}^m}\chi_{A_s}(x) = \sum_i x^{n-2i}(-1)^i m_i$$

*where $m_i$ is the number of matchings (subsets of $E$ that touch each vertex at most once) in $G$ of size $i$.*

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*where $m_i$ is the number of matchings (subsets of $E$ that touch each vertex at most once) in $G$ of size $i$.*

### Proof.

Expand the determinant as a sum over permutations:

1. Permutations that hit any off-diagonal non-edge are $0$
2. Permutations that hit $A_{i,j}$ but not $A_{j,i}$ cancel (in expectation)
3. All that remains are permutations with $n - 2i$ entries on the diagonal and matchings of size $i$

□

# Matching polynomials

For a graph $G$ with $m_i$ matchings of size $i$, the polynomial

$$\mu_G(x) := \sum_i x^{n-2i}(-1)^i m_i$$

is (fittingly) called the *matching polynomial*.

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is (fittingly) called the *matching polynomial*. Some properties:

1. $m_0 = 1$
2. $m_1 = |E|$
3. $m_2$ is the number of pairs of edges that do not share an endpoint
4. $m_{n/2}$ is the number of perfect matchings

In particular, $\mu_G(0)$ is (in general) NP-hard to compute.

# Miracle 2

The matching polynomial was introduced by Heilmann and Lieb in their study of monomers–dimers. In their paper, they prove the somewhat remarkable theorem:

### Theorem (Heilmann–Lieb (1972))

*Let $G$ be a graph with maximum degree $\Delta$. Then*

1. $\mu_G(x)$ *is real-rooted*
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1. $\mu_G(x)$ *is real-rooted*
2. $\mu_G(y) > 0$ *for all* $y > 2\sqrt{\Delta - 1}$

### Proof.

Use the identity

$$\mu_G(x) = \mu_{G-e}(x) - \mu_{G\setminus\{u,v\}}(x)$$

for some $e = (u, v) \in E$ and (a clever) induction. □

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And that is the approach we will take.

# Outline

# In search of a miracle

Unwinding the definition, we get the recurrence equation

$$p_t(x) = \frac{1}{2} p_{t+}(x) + \frac{1}{2} p_{t-}(x)$$

but now we have reached our first major issue.

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In general, it is easy to get the coefficients from the roots but hard to get the roots from the coefficients.

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Approach: forget this and see what we can prove.

# A Lemma

### Lemma

*Let* $f$ *and* $g$ *be monic polynomials. Assume there exists a point* $c \in \mathbb{R}$ *such that* $f$ *and* $g$ *each has* **exactly** *one real root larger than* $c$ *(call these the "extreme roots"). Then the largest real root of* $f + g$ *lies between these extreme roots.*

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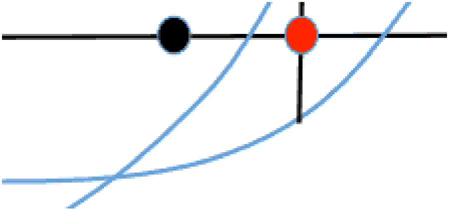
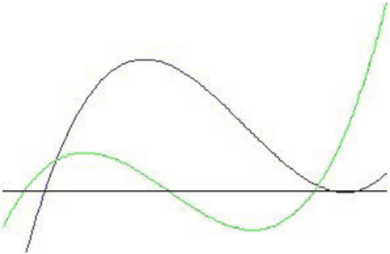
### Proof.

By picture



□

# Without $c$ to "anchor"



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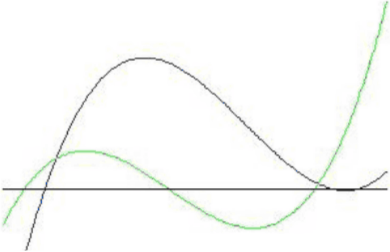

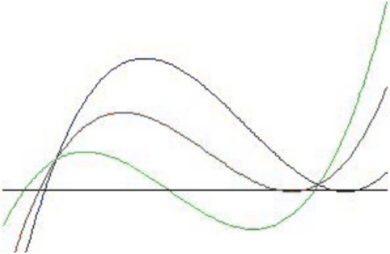

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While each $p_s(x)$ was real-rooted for $s \in \{\pm\}^m$ (characteristic polynomials of symmetric matrices), in general the sums of real-rooted polynomials can be arbitrary.

Example: $p(x) = (x-2)^2 - 1$ (has double root at $1$) and $q(x) = (x+2)^2 - 1$ (has double root at $-1$).

$$p(x) + q(x) = x^2 + 6$$

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But recall that

1. $p_s(x)$ is real-rooted for any $s \in \{\pm\}^m$ (see above)
2. $p_\varnothing(x) = \mu_G(x)$ is real-rooted (by Heilmann and Lieb)

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Perhaps this is true in more generality?

# Equation Revisited

Back to our equation

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Let's worry about $c$ for the moment (keeping real-rootedness on the back burner).

# Interlacing polynomials

Let $p$ be a real-rooted polynomial of degree $n$ and $q$ a real-rooted polynomial of degree $n-1$

$$p(x) = \prod_{i=1}^{n}(x - \alpha_i) \quad \text{and} \quad q(x) = \prod_{i=1}^{n-1}(x - \beta_i)$$

with $\alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_{n-1}$

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We say $q$ *interlaces* $p$ if $\alpha_1 \le \beta_1 \le \alpha_2 \cdots \le \alpha_{d-1} \le \beta_{n-1} \le \alpha_n$.

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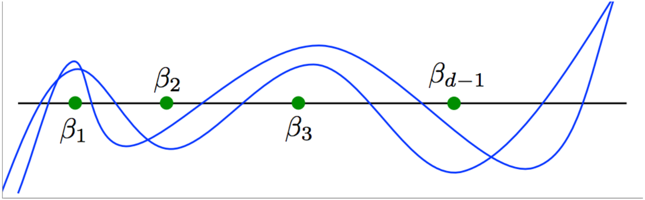
Think: The roots of $q$ separate the roots of $p$

Example 1: $p'(x)$ interlaces $p(x)$
Example 2: If $p$ has no multiple roots (and largest root $R$), then let $q = p/(x-R)$. Then $q(x+\epsilon)$ interlaces $p(x)$

# Common Interlacers

We say that two degree $n$ polynomials $p$ and $r$ have a *common interlacer* if there exists a $q$ such that $q$ interlaces *both* $p$ and $r$ simultaneously.

Think: the roots of $q$ split up $\mathbb{R}$ into $n$ intervals, each of which contains exactly one root of $p$ and one root of $r$



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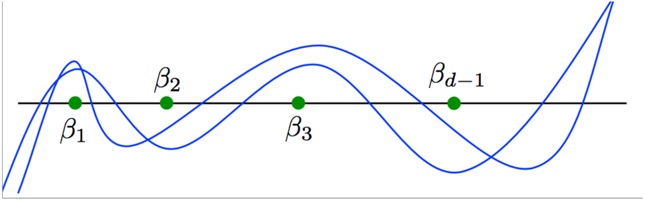


Note, if $p$ and $r$ have a common interlacer (say $q$), then $c = \beta_{d-1}$ can serve as the anchor from the lemma!

# Interlacing families

We say $\{p\}_{s\in\{\pm\}^m}$ is an *interlacing family* if for all partial assignments $t$ we have that

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### Corollary

*If $\{p\}_s$ forms an interlacing family, then there exists an assignment $s^*$ such that the largest root of $p_{s^*}$ is at most the largest root of $p_\varnothing$ (the expected polynomial).*

### Proof.

Start at the expected polynomial and walk backwards. □

# Interlacing for free

Fortunately, interlacing follows directly from a well-known lemma:

Lemma (folklore, Fisk)

*Let $f$, $g$ be polynomials of the same degree such that every $\lambda f + (1 - \lambda) g$ is real-rooted for all $\lambda \in [0, 1]$. Then $f$ and $g$ have a common interlacer.*

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Note this is similar to our recurrence equation:

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So if we can prove that our polynomials are real-rooted for all independent distributions, we get interlacing for free!

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We defined a collection of *partial assignment* polynomials.

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We defined an interlacing family $\{p\}_s$ and showed that any such family has a polynomial $p_{s^*}$ whose largest root is at most the largest root of $p_\emptyset$ (the expected polynomial)

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If we can show the collection of polynomials

$$\mathcal{P} = \left\{ \sum_{s \in \{\pm\}^m} \prod_{s_i = +} \theta_i \prod_{s_i = -} (1 - \theta_i) p_s(x) \;\middle|\; \theta_i \in [0, 1] \right\}$$

are all real-rooted, then our polynomials form an interlacing family.

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$$p_s(x-d) = \chi_{A_s}(x-d) = \det[xI - (dI + A_s)]$$

and that $dI + A_s$ can be written

$$dI + A_s = \sum_{s_i=-}(\delta_i - \delta_j)(\delta_i - \delta_j)^T + \sum_{s_i=+}(\delta_i + \delta_j)(\delta_i + \delta_j)^T$$

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so we can write

$$\sum_{s \in \{\pm\}^m} \prod_{s_i=+} \theta_i \prod_{s_i=-} (1-\theta_i) p_s(x) = \mathbb{E} \det \left[ xI - \sum_{e \in E} \vec{u}_e \vec{u}_e^T \right]$$

where for $e_k = \{i, j\}$

$$\vec{u}_e = \begin{cases} (\delta_i + \delta_j) & \text{with probability } \theta_k \\ (\delta_i - \delta_j) & \text{with probability } 1 - \theta_k \end{cases}$$

# Master Real-rootedness theorem

Thus the real-rootedness of $\mathcal{P}$ would follow from the following theorem:

Theorem

*Let $\vec{u}_1, \ldots, \vec{u}_m$ be* **any** *independent random vectors. Then the expected characteristic polynomial*

$$\mathbb{E}\det\left[xI - \sum_i \vec{u}_i \vec{u}_i^T\right]$$

*is real-rooted.*

# Master Real-rootedness theorem

Thus the real-rootedness of $\mathcal{P}$ would follow from the following theorem:

### Theorem

*Let $\vec{u}_1, \ldots, \vec{u}_m$ be* **any** *independent random vectors. Then the expected characteristic polynomial*

$$\mathbb{E}\det\left[xI - \sum_i \vec{u}_i\vec{u}_i^T\right]$$

*is real-rooted.*

Time to prove some real-rootedness.

# Where to start?

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# Parking garage phenomenon

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Unless you consider them to be a projection of higher dimensional objects.

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A polynomial $p$ is *real stable* if all coefficients are real and $p(z_1, \ldots, z_n) \neq 0$ whenever $\Im(z_i) > 0$ for all $i$ (if $p(z_1, \ldots, z_n) = 0$ then some $z_i$ has $\Im(z_i) \leq 0$).

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Some important properties:

- Univariate polynomials are real-rooted if and only if they are real stable.
- Real stable polynomials are closed under substitution of reals $(z_1, z_2, \ldots, z_n) \to (a, z_2, \ldots, z_n)$ for $a \in \mathbb{R}$.

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Similar to *hyperbolic polynomials*.

# Borcea and Brändén

Borcea and Brändén developed numerous techniques for showing real stability. In particular,

## Lemma

*Let $A_1, \ldots, A_m$ be Hermitian positive semidefinite matrices and $x_1 \ldots x_m$ variables. Then*

$$p(x_1, \ldots, x_m) = \det\left[\sum_{i=1}^{m} x_i A_i\right]$$

*is real stable.*

## Lemma

*If $p(x_1, \ldots, x_m)$ is a real stable polynomial, then*

$$p(x_1, \ldots, x_m) - \frac{\partial p(x_1, \ldots, x_m)}{\partial x_i}$$

*is real stable.*

# Master Identity

We can write the polynomial we want using these operations:

**Theorem**

*Let* $\vec{u}_1, \ldots \vec{u}_m$ *be independent random vectors with* $\mathbb{E}\vec{u}_i\vec{u}_i^T := A_i$*. Then*

$$\mathbb{E}\det\left[xI + \sum_i \vec{u}_i\vec{u}_i^T\right] = \prod_{i=1}^{m}\left(1 - \frac{\partial}{\partial z_i}\right)\det\left[xI + \sum_{i=1}^{m} z_i A_i\right]\Bigg|_{z_1=\cdots=z_m=0}$$

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We call this a *mixed characteristic polynomial* and denote it $\mu[A_1, \ldots, A_m]$.

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**Proof.**

By the first lemma of Borcea and Brändén,

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# Putting it all together, cont.

Since real stability is preserved under substitution by reals, (setting $z_1 = \cdots = z_m = 0$), we have

$$\mu[A_1, \ldots, A_m](x) = \prod_{i=1}^{m} \left(1 - \frac{\partial}{\partial z_i}\right) \det \left[ xI + \sum_{i=1}^{m} z_i A_i \right] \Bigg|_{z_1 = \cdots = z_m = 0}$$

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## Corollary

*Our polynomials form an interlacing family.*

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2. We argued that (for our polynomials) the interlacing condition was implied by the real-rootedness of the partial assignment polynomials (and the recurrence equation)
3. We defined *mixed characteristic polynomials* and showed that our partial assignment polynomials belonged to this class.
4. We showed that mixed characteristic polynomials were real-rooted by using Borcea and Brändén's theory of real stable polynomials.

# Outline

# Piecing things together

## Theorem

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Set $G_{i+1}$ to be the 2-lift associated with $s^*$ — this is bipartite, $d$-regular, and (by Bilu and Linial) Ramanujan — and proceed by induction.

□

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but this would require new proofs of all three stages of the “method of interlacing polynomials”.

1. Showing that the $\{q_s\}$ form an interlacing family
2. Calculating the expected polynomial, and
3. Bounding the largest root of the expected polynomial

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Remember, these things are essentially "everywhere"!

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And thank you for your attention!