Analysis of rank 1 perturbations in general β ensembles

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August 25, 2016

Abstract

We study rank 1 perturbations of matrices where the perturbation vectors are drawn uniformly from the unit ball associated with a general β ensemble. To do this, we use distributions on the unit ball in $\mathbb{R}^n$ derived from the Dirichlet distribution to mimic the behavior of a unit vector drawn uniformly from a general β regime. Our main tool is an identity that expresses certain functions of these random vectors in terms of the Jack symmetric functions. Using this, we extend several properties of random matrices that are well known in the $\beta = \{1, 2, 4\}$ case to general β. We then study the effect of additive rank 1 perturbations drawn from these general β distributions and present some open problems.

1 Introduction

In one of Dyson's most influential papers, he showed that the eigenvalues of random matrices behaved in a way similar to log gases from statistical physics [5]. These gases take on different characteristics as one varies a parameter β which is proportional to the inverse temperature. Dyson showed that he could emulate three particular settings of β using matrix models over different algebraic fields: real symmetric matrices ($\beta = 1$), complex Hermitian matrices ($\beta = 2$), and quaternion self dual matrices ($\beta = 4$), an observation that he called the "threefold way" [6]. Dyson noted that the eigenvalue density functions that he derived from these models generalized naturally to any $\beta > 0$, even without any corresponding matrix models.

Since then, various advances have linked these more general β distributions to various other matrix models, including a collection of tridiagonal models introduced by Edelman et al. [3]. Edelman also suggested a computational scheme he called β-ghosts and conjectured that various properties of random matrices known to exist in the $\beta = \{1, 2, 4\}$ case would also hold in this new scheme [7]. The goal of this paper is to apply a method first introduced by Forrester and Rains [9] to study rank 1 perturbations of random matrices in the general β

*Research supported by NSF CAREER grant 1552520.

regime. In [9], Forrester and Rains observed that when a random real vector was drawn using a Dirichlet distribution, it behaved as though it were drawn from a uniform distribution in the general β regime (in certain circumstances). Using this idea, we show that a number of properties of random matrices that are known to exist in the $\beta = \{1, 2, 4\}$ cases extend to the general β case, providing evidence for some of Edelman's conjectures.

Our main analytic tool for studying these Dirichlet weighted random vectors (Theorem 3.5) is an expression of certain weighted moments in terms of the Jack symmetric functions, a collection of symmetric functions with deep connections to random matrices in the $\beta = \{1, 2, 4\}$ regime (see Section 2.4). In particular, we show that an identity that characterizes the Haar distribution in the $\beta = \{1, 2, 4\}$ cases holds in general for these Dirichlet weighted random vectors [11]. We then use this tool to study the effect of rank 1 (additive) perturbations of matrices when the perturbation is drawn from a general β distribution.

1.1 Organization

The paper is organized as follows: we introduce some of the preliminary notions that we will use in the paper in Section 2, including a short review of the Dirichlet distribution (Section 2.3) and Jack symmetric functions (Section 2.4). In Section 3, we introduce the measures from [9] based on the Dirichlet distribution and show that they extend some fundamental identities (in the $\beta = \{1, 2, 4\}$ regimes) involving the Jack symmetric functions to general β. In Section 3.2, we move onto the situation of rank 1 perturbations of matrices. We prove two formulas concerning the structure of the expansions of rank 1 perturbations of matrices in terms of Jack symmetric functions. We also give some evidence (placed in an appendix, due to its unnecessary amount of computation) that the formulas are unique to these distributions (as far as sufficiently symmetric distributions are concerned). We end with a discussion of further research directions and open problems.

2 Preliminaries

2.1 Abuse of notation

The majority of this paper will be concerned with the evaluation of symmetric functions on the eigenvalues of matrices. Accordingly, we will abuse notation: for a symmetric function f and an $n \times n$ matrix A, we will simply write $f(A)$ to denote the function f evaluated at the *eigenvalues* of A. We will maintain the convention of using capital letters for matrices and lower case letters for vectors. In other words, for a matrix A, the translation

$$\boxed{f(A) := f(\text{eigen}(A))}$$

will hold throughout this paper.

2.2 Generalized binomials

For $r \in \mathbb{R}$ and $k \in \mathbb{N}$, the *generalized binomial coefficients* are defined as

$$\binom{r}{k} = \frac{\Gamma(r+1)}{\Gamma(r-k+1)\, k!}$$

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \tag{1}$$

is the usual *Gamma function*. We will make frequent use of two well known identities. The first is sometimes known as the *generalized binomial theorem*:

$$(1+x)^r = \sum_{i=0}^{\infty} \binom{r}{i} x^i. \tag{2}$$

and the second follows directly from the definition:

$$\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}. \tag{3}$$

For nonnegative integers $t_1, \ldots, t_n$ with $\sum_{i=1}^n t_i = k$, the *multinomial coefficients* are defined as

$$\binom{k}{t_1, \ldots, t_n} = \frac{k!}{t_1! t_2! \ldots t_n!}.$$

To avoid notational confusion, we will only use real numbers in the binomial case (that is, when $n = 2$). In all other cases, we will use only nonnegative integers, and so standard factorials can be used.

2.3 Dirichlet distributions

For real numbers $a_1, \ldots, a_n > 0$, the *Dirichlet distribution* $\mathrm{Dir}(a_1, \ldots, a_n)$ is the multivariate distribution with density function (with respect to Lebesgue measure)

$$f(x_1, \ldots, x_n; a_1, \ldots, a_n) \propto x_1^{a_1-1} x_2^{a_2-1} \ldots x_n^{a_n-1}$$

which has support on the n-dimensional simplex

$$\Delta_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \sum_i x_i = 1 \right\}.$$

Because it is supported on the simplex, one can think of the vector $(x_1, \ldots, x_n)$ as being a discrete probability distribution; the Dirichlet distribution then acts as a "distribution on distributions." Because of this, the Dirichlet distribution is widely used in Bayesian inference as a prior distribution [1]. In such a scenario, the parameters $a_1, \ldots, a_n$ are typically known as "concentration" parameters, since large parameters result in the inferred distribution being spread out among the n possibilities whereas small parameters tends to create distributions that are concentrated over a small subset of the possibilities.

For our purposes, much of the utility of the distribution will lie in its normalization, which we denote $B(a_1, \dots, a_n)$:

$$B(a_1, \dots, a_n) = \int_{\Delta_n} x_1^{a_1-1} x_2^{a_2-1} \dots x_n^{a_n-1} d\vec{x} = \frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_n)}{\Gamma(a_1 + \dots + a_n)}. \tag{4}$$

Note that (4) can be seen as a generalization of the *Beta function*

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

to larger dimensions (and is the reason for our use of the letter B for the normalization).

2.4 Jack symmetric functions

To introduce the Jack symmetric functions, we must first introduce some notions from the theory of partitions.

2.4.1 Partitions

For a positive integer n, we will say that a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ is a *partition of* n if

1. $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$
2. $\sum_i \lambda_i = n$

and we will write $|\lambda| = n$. We will let Λ_n denote the collection of partitions of n and $\Lambda_* = \bigcup_n \Lambda_n$. The *length* of a partition, written $\ell(\lambda)$, is the largest k for which $\lambda_k \neq 0$ (it should be clear from the definition that only a finite number of elements of λ_i can be nonzero, and that they must occupy an initial interval of λ). For ease of reading, we will use the customary exponential notation: that is, we will write the partition

$$(\underbrace{t_1, \dots, t_1}_{n_1 \text{times}}, \underbrace{t_2, \dots, t_2}_{n_2 \text{times}}, \dots)$$

as $(t_1^{n_1}, t_2^{n_2}, \dots)$. The one exception will be the values of 0, which we will never include in any presentation unless necessary (but which will always exist).

Given two partitions of n, (say λ and μ), we say that μ *dominates* λ, written $\lambda \preceq \mu$, if

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \tag{5}$$

for all integers k. In the case that (5) holds for partitions λ, μ with $|\lambda| < |\mu|$, we will write $\lambda \preceq_w \mu$ and say μ *weakly dominates* λ. Both domination and weak domination define partial orders on the collection of partitions known as the *dominance ordering* and *weak dominance ordering*, respectively.

Remark 2.1. The name "weak dominance" comes from the fact that when $|\lambda| = |\mu|$, the inequality in (5) will become an equality for all $k > \max\{\ell(\lambda), \ell(\mu)\}$ and is therefore considered to be a "stronger" condition.

Recall that a *Young diagram* is a finite collection of boxes (called cells) which are arranged in left-justified rows with nonincreasing lengths. There is a natural bijection between partitions and Young diagrams where the ith row of the Young diagram has λ_i cells in it. There is a natural conjugation operation on Young diagram which entails switching the rows and columns (similar to a matrix transpose). The corresponding partition is known as the *conjugate* partition and will be denoted λ'. Letting $N_i(\lambda)$ denote the number of times that i appears in λ, one can equivalently define λ' as

$$\lambda'_i = \sum_{t \geq i} N_t(\lambda).$$

One straightforward lemma we will use is that the conjugate operation is order reversing with respect to the dominance order [2]:

Lemma 2.2. *For all partitions λ, μ with $|\lambda| = |\mu|$, we have*

$$\lambda \preceq \mu \iff \mu' \preceq \lambda' \tag{6}$$

We warn that the analogous version of (6) for weak dominance is *not* true in general, as can be seen by the simple counterexample $\lambda = (1)$ and $\mu = (2^2)$, as $\lambda \preceq_w \mu$ and both λ and μ are self-conjugate.

2.4.2 Jack Symmetric Functions

Let $\vec{x} = x_1, x_2, \ldots$ be a sequence of formal variables. A function $p(\vec{x})$ is called *symmetric* if it is invariant under permutation of its variables. It is called *homogeneous* of degree n if $p(c\vec{x}) = c^n p(\vec{x})$. Given a finite collection of variables $y_1, \ldots, y_n$, we will write $p(y_1, \ldots, y_n)$ to denote the evaluation of p on the sequence $(y_1, \ldots, y_n, 0, 0, \ldots)$. The restriction of a symmetric, homogeneous function to a finite number of nonzero entries is a (multivariate) polynomial in its nonzero entries. Therefore, one does not need to worry about convergence in such situations (which will be the only situations treated in this paper).

Because partitions represent all ways to total a given integer, they are a natural indexing set for symmetric homogeneous functions. There are well-known collections of functions, indexed by partitions, that form a basis for the space of the symmetric functions. We will utilize two such bases, the first of which are the *power-sum functions*

$$p_\lambda(\vec{x}) = \prod_{i=1}^{\ell(\lambda)} \left(\sum_j x_j^{\lambda_i} \right)$$

and the second of which are the *monomial symmetric functions*, defined as the sum of all monomials whose exponents (when put into increasing order) form the partition λ. In both cases, we find examples to be more easily understood than the definitions:

Example 2.3. For variables $(w, x, y, z, \dots)$, we have

- $p_{(2,1^2)}(w, x, y, z, \dots) = (w + x + y + z + \dots)^2(w^2 + x^2 + y^2 + z^2 + \dots)$
- $m_{(2,1^2)}(w, x, y, z, \dots) = wxy^2 + wxz^2 + w^2xy + w^2xz + wyz^2 + wx^2z + \dots$

The power-sum and monomial symmetric functions are connected in a natural way by multinomial expansions:

$$p_{(1^k)}(\vec{x}) = \left(\sum_i x_i\right)^k = \sum_{|\lambda|=k} \binom{k}{\lambda_1, \dots, \lambda_n} m_\lambda(\vec{x}) \tag{7}$$

where

$$\binom{k}{\lambda_1, \dots, \lambda_n} = \frac{k!}{\lambda_1! \dots \lambda_n!}$$

is a multinomial coefficient defined in Section 2.2.

Since the power-sum functions form a basis, one can define an inner product on the space of symmetric functions simply by defining its values on these elements (and extending linearly). One such inner product can be seen as a weighted version of the Hall inner product [11]: for $\alpha > 0$,

$$\langle p_\lambda(\vec{x}), p_\mu(\vec{x})\rangle_\alpha = \delta_{\lambda\mu}\alpha^{\ell(\lambda)} \prod_{i=1}^{\infty} i^{N_i(\lambda)} N_i(\lambda)! \tag{8}$$

where $\delta_{\lambda,\mu}$ is 1 whenever $\lambda = \mu$ and 0 otherwise (the function $N_i(\lambda)$ is defined in Section 2.4.1). A fundamental result of Macdonald [11] gives a second explicit orthogonal basis for each such inner product:

Theorem 2.4. *For any $\alpha > 0$, there exists a unique collection of symmetric functions $\{J_\lambda(\vec{x}; \alpha)\}_{\lambda\in\Lambda}$ that satisfy the following conditions:*

Orthogonality: $\langle J_\lambda(\vec{x}; \alpha), J_\mu(\vec{x}; \alpha)\rangle_\alpha = 0$ *whenever* $\lambda \neq \mu$,

Triangularity: *There exist coefficients $v_{\lambda,\mu}(\alpha)$ such that*

$$J_\lambda(\vec{x}; \alpha) = \sum_{\substack{\mu:|\mu|=|\lambda| \\ \mu\preceq\lambda}} v_{\lambda,\mu}(\alpha) m_\mu(\vec{x}),$$

Normalization: $v_{\lambda,1^n}(\alpha) = n!$ *whenever* $|\lambda| = n$.

The functions $J_\lambda(\vec{x}; \alpha)$ satisfying Theorem 2.4 are known as the *Jack symmetric functions*. We make two remarks:

Remark 2.5.

1. Since the triangularity property only includes terms with $|\mu| = |\lambda|$, $J_\lambda(\vec{x};\alpha)$ is a symmetric, homogeneous function of degree $|\lambda|$.

2. It is easy to see that $m_\lambda(y_1,\ldots,y_n) = 0$ whenever $n < \ell(\lambda)$. Since $\mu \preceq \lambda$ requires $\ell(\mu) \geq \ell(\lambda)$, the triangularity property implies that $J_\lambda(y_1,\ldots,y_n;\alpha) = 0$ whenever $n < \ell(\lambda)$.

A number of different normalizations for the Jack symmetric functions appear in the literature (for example, the C and P normalizations [4]). Rather than describe these normalizations, we will define our own (which is easily obtained from any of the others): for a partition λ, we define

$$\widehat{J}_\lambda(x_1,\ldots,x_n;\alpha) = \begin{cases} \frac{J_\lambda(x_1,\ldots,x_n;\alpha)}{J_\lambda(1^n;\alpha)} & \text{for } \ell(\lambda) \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

One of the first papers to treat the combinatorial properties of Jack symmetric functions was due to Stanley [13]. We end the section by recalling two results from this paper. The first gives an explicit formulas for the Jack symmetric functions when the partition has the special form $\lambda = (k)$:

Lemma 2.6. *For $\alpha > 0$,*

$$\frac{(-\alpha)^k}{k!} J_{(k)}(\vec{x};1/\alpha) = \sum_{|\lambda|=k} \left(\prod_i \binom{-\alpha}{\lambda_i} \right) m_\lambda(\vec{x}). \tag{10}$$

In particular,

$$\frac{(-\alpha)^k}{k!} J_{(k)}(1^n;1/\alpha) = \binom{-n\alpha}{k}. \tag{11}$$

The second result gives a generating function for Jack symmetric functions of the same type. Since the normalization used here differs from [13], we include a short proof:

Lemma 2.7. *For all $n \times n$ matrices Y and all $\alpha > 0$, we have*

$$\det\left[I + Y\right]^{-\alpha} = \sum_k \binom{-n\alpha}{k} \widehat{J}_{(k)}(Y;1/\alpha)$$

Proof. Let Y have eigenvalues $y_1,\ldots,y_n$. Then

$$\det\left[I + Y\right]^{\alpha} = \prod_i (1+y_i)^{-\alpha} \overset{(2)}{=} \prod_i \sum_{j_i} \binom{-\alpha}{j_i} y_i^{j_i} = \sum_{j_1,\ldots,j_n} \binom{-\alpha}{j_1} \cdots \binom{-\alpha}{j_n} y_1^{j_1} \cdots y_n^{j_n}.$$

where, if we group the monomials by the partition formed by their exponents, we get

$$= \sum_{j_1,\ldots,j_n} \binom{-\alpha}{j_1} \cdots \binom{-\alpha}{j_n} y_1^{j_1} \ldots y_n^{j_n}. = \sum_k \sum_{|\lambda|=k} \left(\prod_{i=1}^{\ell(\lambda)} \binom{-\alpha}{\lambda_i} \right) m_\lambda(Y).$$

On the other hand, Lemma 2.6 implies

$$\binom{-n\alpha}{k} \widehat{J}_{(k)}(Y; 1/\alpha) \stackrel{(11)}{=} \frac{(-\alpha)^k}{k!} J_{(k)}(Y; 1/\alpha) \stackrel{(10)}{=} \sum_{|\lambda|=k} \left(\prod_{i=1}^{\ell(\lambda)} \binom{-\alpha}{\lambda_i} \right) m_\lambda(Y)$$

and so the result follows by summing over k. $\square$

We will discuss further properties of the Jack symmetric functions (in particular, as they apply to random matrices) in Section 3.1. We refer the interested reader to [11] and [13] for more comprehensive and combinatorial introductions (respectively).

3 Dirichlet-weighted random vectors

Definition 3.1. We will call the distribution on unit vectors $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with probability density function

$$f(x_1, \ldots, x_n) \propto |x_1 x_2 \ldots x_n|^{\alpha-1} \tag{12}$$

the *unit α distribution*, written μ_n^α.

The μ_n^α distribution can be directly related to the Dirichlet distribution (see Section 2.3) by a simple change of variables:

Lemma 3.2. *Let $s_1, \ldots s_n$ be nonnegative integers and let $t_1, \ldots, t_n$ be positive real numbers. Then*

$$\int_{S^{n-1}} x_1^{s_1} \ldots x_n^{s_n} |x_1^{t_1-1} \ldots x_n^{t_n-1}| d\vec{x} = \begin{cases} B\left(\frac{s_1+t_1}{2}, \ldots, \frac{s_n+t_n}{2}\right) & \textit{if all } s_i \textit{ are even} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. The case when any s_i is odd is easy to see, since then the values at $(x_1, \ldots, x_i, \ldots, x_n)$ and $(x_1, \ldots, -x_i, \ldots, x_n)$ will cancel each other out. For the case when all s_i are even, each of the 2^n orthants will give the same values and so we can restrict the integral to the nonnegative orthant and multiply the result by 2^n. Hence we have

$$\int_{S^{n-1}} x_1^{s_1} \ldots x_n^{s_n} |x_1^{t_1-1} \ldots x_n^{t_n-1}| d\vec{x} = 2^n \int_{S_+^{n-1}} x_1^{s_1+t_1-1} \ldots x_n^{s_n+t_n-1} d\vec{x}$$

where $S_+^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \sum x_i^2 = 1\}$. Now we make the substitution $u_i = x_i^2$ so that $2x_i dx_i = du_i$, or equivalently, $dx_i = \frac{1}{2}(u_i)^{-1/2} du_i$. Hence we get

$$\begin{aligned}
2^n \int_{S_+^{n-1}} x_1^{s_1} \ldots x_n^{s_n} |x_1^{t_1} \ldots x_n^{t_n}| d\vec{x} &= \int_{\Delta_n} u_1^{(s_1+t_1-1)/2} \ldots u_n^{(s_n+t_n-1)/2} (u_1 u_2 \ldots u_n)^{-1/2} d\vec{u} \\
&= \int_{\Delta_n} u_1^{(s_1+t_1)/2-1} \ldots u_n^{(s_n+t_n)/2-1} d\vec{u} \\
&= B\left(\frac{s_1+t_1}{2}, \ldots, \frac{s_n+t_n}{2}\right)
\end{aligned}$$

with the last equality coming from (4). □

In particular, Lemma 3.2 implies that the normalization factor in (12) is

$$\int_{S^{n-1}} |x_1 x_2 \ldots x_n|^{\alpha-1} d\vec{x} = B(\alpha/2, \ldots, \alpha/2).$$

Similar to the Dirichlet distribution, we can think of the parameter α as determining how "concentrated" the mass of the vector will be in a few coordinates. For example,

1. μ_n^1 is the Haar measure on S^{n-1}.
2. μ_n^0 is the discrete measure on the collection $\{\pm e_i\}$ (where e_i are the standard basis vectors)
3. μ_n^∞ is the discrete measure on the collection $\{\frac{1}{\sqrt{n}}(\pm 1, \pm 1, \ldots, \pm 1)\}$.

Calculations related to these distributions inevitably entail an application of Lemma 3.2:

Corollary 3.3. *Let $p(\vec{x}) = x_1^{s_1} \ldots x_n^{s_n}$ be a monomial and let $v \sim \mu_n^{2\alpha}$. Then*

$$\mathbb{E}_v\{p(v)\} = \frac{B\left(\alpha + \frac{s_1}{2}, \ldots, \alpha + \frac{s_n}{2}\right)}{B(\alpha, \ldots, \alpha)}$$

when all s_i are even (and 0 otherwise).

3.1 Jack Formulas

In this section, we prove our main technical tool (Theorem 3.5) and then show that the distribution μ_n^α defined in the previous section satsifies an important relationship between Jack symmetric functions and random matrices (Corollary 3.7). We begin by recording an observation that will be used multiple times in the forthcoming proofs:

Lemma 3.4. *Let X be an $n \times n$ matrix with rank 1. Then*

$$\widehat{J}_\lambda(X; 1/\alpha) = \begin{cases} \mathrm{Tr}\left[X^k\right] \frac{\binom{-\alpha}{k}}{\binom{-n\alpha}{k}} & \textit{for } \lambda = (k) \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We first note that, since X has only one nonzero eigenvalue, Remark 2.5 implies that $\widehat{J}_\lambda(X : 1/\alpha) = 0$ whenever $\ell(\lambda) > 1$, and so it remains to prove the case $\lambda = (k)$. Let w be the single nonzero eigenvalue. Since J_λ is homogeneous, we have

$$\widehat{J}_{(k)}(X; 1/\alpha) = w^k \widehat{J}_{(k)}(1; 1/\alpha) = \operatorname{Tr}\left[X^k\right] \widehat{J}_\lambda(1; 1/\alpha).$$

Hence it suffices to prove the lemma in the case when $w = 1$, for which we have

$$\widehat{J}_\lambda(1; 1/\alpha) = \frac{J_{(k)}(1; 1/\alpha)}{J_{(k)}(1^n; 1/\alpha)} \overset{(11)}{=} \frac{\binom{-\alpha}{k}}{\binom{-n\alpha}{k}}$$

by Lemma 2.6. □

We now prove our main technical tool, which we state as a theorem:

Theorem 3.5. *If A is a diagonal matrix and $v \sim \mu_n^{2\alpha}$ then*

$$\mathbb{E}_v\left\{(v^T A v)^k\right\} = \widehat{J}_{(k)}(A; 1/\alpha)$$

for all integers k.

Proof. Let $a_1, \ldots, a_n$ be the diagonal entries of A. Since A is diagonal, we have $v^T A v = \sum_i v_i^2 a_i$ and so by (7), we can write

$$\mathbb{E}_v\left\{(v^T A v)^k\right\} = \sum_{|\lambda|=k} \binom{k}{\lambda_1, \ldots, \lambda_n} \mathbb{E}_v\left\{m_\lambda(v_1^2 a_1, \ldots, v_n^2 a_n)\right\}.$$

We now use Corollary 3.3: since $B(\vec{x})$ is a symmetric function, each monomial in a fixed m_λ is going to give the same value under the expectation. Hence Corollary 3.3 implies

$$\mathbb{E}_v\left\{m_\lambda(v_1^2 a_1, \ldots, v_n^2 a_n)\right\} = m_\lambda(a_1, \ldots, a_n) \frac{B(\alpha + \lambda_1, \alpha + \lambda_2, \ldots, \alpha + \lambda_n)}{B(\alpha, \ldots, \alpha)}$$

and so we get

$$\begin{aligned}
\mathbb{E}_v\left\{(v^T A v)^k\right\} &= \sum_{|\lambda|=k} \binom{k}{\lambda_1, \ldots, \lambda_n} \frac{B(\alpha + \lambda_1, \alpha + \lambda_2, \ldots, \alpha + \lambda_n)}{B(\alpha, \ldots, \alpha)} m_\lambda(a_1, \ldots a_n) \\
&= \sum_{|\lambda|=k} \binom{k}{\lambda_1, \ldots, \lambda_n} \frac{\Gamma(n\alpha)}{\Gamma(n\alpha + k)} \left(\prod_{i=1}^{\ell(\lambda)} \frac{\Gamma(\alpha + \lambda_i)}{\Gamma(\alpha)} \right) m_\lambda(a_1, \ldots a_n) \\
&\overset{(3)}{=} \frac{1}{\binom{-n\alpha}{k}} \sum_{|\lambda|=k} \left(\prod_{i=1}^{\ell(\lambda)} \binom{-\alpha}{\lambda_i} \right) m_\lambda(a_1, \ldots a_n) \\
&\overset{(10)}{=} \widehat{J}_{(k)}(A; 1/\alpha)
\end{aligned}$$

where the last identity comes from Lemma 2.6. □

The Jack symmetric functions have an intimate relationship with classical β ensembles due to their role as *zonal polynomials* [8, Section 13.4.3].

Lemma 3.6. *For $\beta \in \{1,2,4\}$, let A and B be real ($\beta = 1$), complex ($\beta = 2$) or quaternionic ($\beta = 4$) matrices and let U be a uniformly distributed unitary matrix over the same field (and $\dagger$ the corresponding adjoint operator). Then*

$$\mathbb{E}\left\{\widehat{J}_\lambda(AU^\dagger BU; 2/\beta)\right\} = \widehat{J}_\lambda(A; 2/\beta)\widehat{J}_\lambda(B; 2/\beta)$$

for all partitions λ.

One of the issues with extending Lemma 3.6 to the case of general β is the lack of an obvious analogue for a "uniformly distributed unitary matrix" (though various proposals have been made, e.g. [7]). Our first application of Theorem 3.5 shows that, at least as far as random vectors are concerned, the $\mu_n^{2\alpha}$ distribution gives the appropriate generalization of Lemma 3.6.

Corollary 3.7. *If A is a diagonal matrix and $v \sim \mu_n^{2\alpha}$, then*

$$\mathbb{E}_v\left\{\widehat{J}_\lambda(Avv^T; 1/\alpha)\right\} = \widehat{J}_\lambda(A; 1/\alpha)\widehat{J}_\lambda(e_1e_1^T; 1/\alpha)$$

for all partitions λ.

Before giving the proof, we remark that, by Lemma 3.4, the value of $\widehat{J}_\lambda(ww^T; 1/\alpha)$ is a function of $\|w\|$ only (not the direction), and so the use of the vector e_1 in the statement of Corollary 3.7 is arbitrary (any unit vector would be equivalent). One could, in fact, use $\widehat{J}_\lambda(vv^T; 1/\alpha)$, but we chose to use a fixed unit vector so as to avoid any confusion as to what is truly "random".

Proof. We start by noticing that if $\ell(\lambda) > 1$, then by Lemma 3.4,

$$\widehat{J}_\lambda(Avv^T; 1/\alpha) = \widehat{J}_\lambda(e_1e_1^T; 1/\alpha) = 0$$

since both Avv^T and vv^T have rank 1. It then remains to prove the case when $\lambda = (k)$ for some integer $k > 0$. For a general vector w, Lemma 3.4 implies

$$\widehat{J}_\lambda(Aww^T; 1/\alpha) = \mathrm{Tr}\left[(Aww^T)^k\right]\frac{\binom{-\alpha}{k}}{\binom{-n\alpha}{k}} = (w^TAw)^k\widehat{J}(e_1e_1^T).$$

Taking the expectation over v and applying Theorem 3.5 then completes the proof. $\square$

We end this section with two applications of Theorem 3.5 that will prove useful in subsequent sections. The first application is to the generating function in Lemma 2.7:

Corollary 3.8. *If Y is a diagonal $n \times n$ matrix and $v \sim \mu_n^{2\alpha}$, then*

$$\det[Y]^{-\alpha} = \mathbb{E}_v\left\{(v^TYv)^{-n\alpha}\right\}$$

Proof. By Lemma 2.7 and Theorem 3.5, we have

$$\det[Y]^{-\alpha} = \sum_k \binom{-n\alpha}{k} \widehat{J}_{(k)}(Y - I; 1/\alpha) = \sum_k \binom{-n\alpha}{k} \mathbb{E}_v \left\{ \left(v^T(Y-I)v\right)^k \right\}$$

where

$$\sum_k \binom{-n\alpha}{k} \mathbb{E}_v \left\{ \left(v^T(Y-I)v\right)^k \right\} = \mathbb{E}_v \left\{ (1 + v^T(Y-I)v)^{-n\alpha} \right\} \overset{(2)}{=} \mathbb{E}_v \left\{ (v^T Y v)^{-n\alpha} \right\}.$$

□

The second application is an identity:

Corollary 3.9. *Let $w \sim \mu_n^{2\alpha}$. Then*

$$(1-s)^{-\alpha} = \mathbb{E}_w \left\{ (1 - s w^T e e^T w)^{-n\alpha} \right\}$$

Proof. Note that, by (2) we have

$$(1-s)^{-\alpha} = \sum_i \binom{-\alpha}{i} (-s)^i$$

and

$$\mathbb{E}_w \left\{ (1 - s w^T e e^T w)^{-n\alpha} \right\} = \sum_i \binom{-n\alpha}{i} \mathbb{E}_w \left\{ (w^T e e^T w)^i \right\} (-s)^i.$$

Plugging in

$$\mathbb{E}_w \left\{ (w^T e e^T w)^i \right\} = \frac{\binom{-\alpha}{i}}{\binom{-n\alpha}{i}},$$

which follows from Theorem 3.5, finishes the lemma. □

3.2 Rank 1 perturbations

The goal of this section is to establish formulas for rank 1 perturbations of matrices when the perturbing vectors have the $\mu_n^{2\alpha}$ distribution. We start by establishing a technical lemma that will do much of the work for us. For this section we will use e to denote an (arbitrary) standard basis vector.

Lemma 3.10. *Let $u, v, w \sim \mu_n^{2\alpha}$ be independent for $\alpha > 0$. Then*

$$\mathbb{E}_v \left\{ \det\left[I - A - t v v^T\right]^{-\alpha} \right\} = \mathbb{E}_{w,u} \left\{ (1 - t w^t e e^T w - u^T A u)^{-n\alpha} \right\}$$

holds for all $n \times n$ diagonal matrices A.

Proof. We first show the case when $I - A$ is invertible. Recall that the *matrix determinant lemma* [10] asserts that

$$\det\left[I - A - tvv^T\right]^{-\alpha} = \det\left[I - A\right]^{-\alpha}\left(1 - tv^T(I-A)^{-1}v\right)^{-\alpha}.$$

and applying Corollary 3.9 with $s = tv^T(I-A)^{-1}v$ then gives

$$\mathbb{E}_v\left\{\det\left[I - A - tvv^T\right]^{-\alpha}\right\} = \det\left[I - A\right]^{-\alpha}\mathbb{E}_{w,v}\left\{\left(1 - (tw^Tee^Tw)v^T(I-A)^{-1}v\right)^{-n\alpha}\right\}.$$

where, since $I - (tw^Tee^Tw)(I-A)^{-1}$ is diagonal,

$$\mathbb{E}_v\left\{\left(1 - (tw^Tee^Tw)v^T(I-A)^{-1}v\right)^{-n\alpha}\right\} = \det\left[I - (tw^Tee^Tw)(I-A)^{-1}\right]^{-\alpha}$$

by Corollary 3.8. Hence we have

$$\begin{aligned}\mathbb{E}_v\left\{\det\left[I - A - tvv^T\right]^{-\alpha}\right\} &= \det\left[I - A\right]^{-\alpha}\mathbb{E}_w\left\{\det\left[I - (tw^Tee^Tw)(I-A)^{-1}\right]^{-\alpha}\right\}\\ &= \mathbb{E}_w\left\{\det\left[(1 - tw^Tee^Tw)I - A\right]^{-\alpha}\right\}.\end{aligned}$$

Now since $(1 - tw^Tee^Tw)I - A$ is diagonal (for any w), we can use Corollary 3.8 yet again to write

$$\begin{aligned}\det\left[(1 - tw^Tee^Tw)I - A\right]^{-\alpha} &= \mathbb{E}_u\left\{\left(u^T\left((1 - tw^Tee^Tw)I - A\right)u\right)^{-n\alpha}\right\}\\ &= \mathbb{E}_u\left\{(1 - tw^Tee^Tw - u^TAu)^{-n\alpha}\right\},\end{aligned}$$

proving the lemma whenever $I - A$ is invertible. Now for the case when $I - A$ is not invertible, we simply apply the previous argument to the (cofinite) collection of points x for which $I - xA$ is invertible and extend by continuity. □

The proof of Lemma 3.10 shows the utility of identities like the one in Corollary 3.8 (and, in turn, Theorem 3.5). By replacing a determinant with an expected power, one gains some amount of flexibility that is not obvious in the original determinant. We can now use Lemma 3.10 to derive an identity for the probability distributions of rank 1 perturbations.

Theorem 3.11. *Let A be an $n \times n$ diagonal matrix, and let $v, w \sim \mu_n^{2\alpha}$ be independent random vectors. For $t \in \mathbb{R}$, let $B(A, v, t)$ be the (random) diagonal matrix with entries that are the (random) eigenvalues of $A + tvv^T$. Then*

$$w^TB(A,v,t)w \qquad \textit{and} \qquad w^TAw + tv^Tee^Tv$$

have the same distributions.

Proof. Consider the generating functions

$$p_{A,t}(x) = \sum_k \binom{-n\alpha}{k}(-x)^k\mathbb{E}_{w,v}\left\{(w^TB(A,v,t)w)^k\right\}$$

and

$$q_{A,t}(x) = \sum_k \binom{-n\alpha}{k} (-x)^k \mathbb{E}_{w,v} \left\{ (w^T A w + t v^T e e^T v)^k \right\}.$$

Then we can write

$$\begin{aligned}
p_{A,t}(x) &= \sum_k \binom{-n\alpha}{k} (-x)^k \mathbb{E}_{w,v} \left\{ (w^T B(A, v, t) w)^k \right\} \\
&= \sum_k \binom{-n\alpha}{k} (-x)^k \mathbb{E}_v \left\{ \widehat{J}_{(k)}(B(a, v, t); 1/\alpha) \right\} && \text{(Theorem 3.5)} \\
&= \mathbb{E}_v \left\{ \det \left[I - xB(a, v, t) \right]^{-\alpha} \right\} && \text{(Lemma 2.7)} \\
&= \mathbb{E}_v \left\{ \det \left[I - x(A - tvv^T) \right]^{-\alpha} \right\} && (*) \\
&= \mathbb{E}_{w,v} \left\{ (1 - xtw^t e e^T w - x v^T A v)^{-n\alpha} \right\} && \text{(Lemma 3.10)} \\
&\overset{(2)}{=} \sum_k \binom{-n\alpha}{k} (-x)^k \mathbb{E}_{w,v} \left\{ (tw^t e e^T w + v^T A v)^k \right\} \\
&= q_{A,t}(x)
\end{aligned}$$

where the step marked $(*)$ is due to the unitary invariance of the determinant. Since two power series are the same if and only if they have the same coefficients, this implies

$$\mathbb{E}_{w,v} \left\{ (tw^t e e^T w + v^T A v)^k \right\} = \mathbb{E}_{w,v} \left\{ (w^T B(A, v, t) w)^k \right\}$$

for all k. Since two distributions that have equal (and finite) moments are the same, this implies the theorem. □

We note that Theorem 3.11 is (by Theorem 3.5) equivalent to the statement

$$\mathbb{E}_v \left\{ \widehat{J}_{(k)}(A + vv^T; 1/\alpha) \right\} = \sum_{i=0}^{k} \binom{k}{i} \widehat{J}_{(i)}(ee^T; 1/\alpha) \widehat{J}_{(k-i)}(A; 1/\alpha) \tag{13}$$

for $v \sim \mu_n^{2\alpha}$. One might ask if a similar identity holds for more general partitions λ (of length longer than 1). To investigate this, we employ yet another result of Stanley showing that the collection of products of the Jack symmetric functions with partitions of length 1 form a basis for the symmetric functions [13].

Definition 3.12. For a partition λ, let $\mathcal{J}_\lambda(\vec{x}, \alpha)$ be the symmetric function

$$\mathcal{J}_\lambda(\vec{x}; \alpha) = \prod_{i=1}^{\ell(\lambda)} J_{(\lambda_i)}(A).$$

In [13], Stanley showed the following relationship:

Theorem 3.13. *For all partitions λ with $|\lambda| = k$,*

$$\mathcal{J}_\lambda(\vec{x};\alpha) = \sum_{|\mu|=k} v_{\mu,\lambda} \frac{\alpha^k \prod_i \lambda_i!}{j_\lambda} J_\mu(\vec{x};\alpha)$$

where the coefficients $v_{s,t}$ are the same as the ones in Theorem 2.4 and

$$j_\lambda = \langle J_\lambda(x;\alpha), J_\lambda(x;\alpha)\rangle_\alpha$$

is the normalization constant from the inner product defined in (8).

Note that, although the coefficients $v_{s,t}$ are the same as those in Theorem 2.4, the usage in Theorem 3.13 is somewhat different — if we let V be the matrix with $V(\lambda,\mu) = v_{\lambda,\mu}$, then Theorem 3.13 asserts that the transformation between the $\mathcal{J}$ functions and the normal Jack functions uses the matrix V^T. In particular, since V is upper triangular, the transformation in Theorem 3.13 will be lower triangular. This leads to the following observation (see Section 2.4.1 for the definitions of the dominance orderings):

Theorem 3.14. *Let A be a diagonal matrix and let $v \sim \mu_n^{2/\alpha}$. Then*

$$\mathbb{E}_v \left\{ \left\langle J_\lambda(A + vv^T;\alpha), J_\lambda(A;\alpha)\right\rangle_\alpha \right\} \neq 0 \tag{14}$$

implies $\tau' \preceq_w \lambda'$.

Proof. Let $U = V^{-1}$ where $V = V(\lambda,\mu)$ is the matrix defined above. Then by Theorem 3.11, we have

$$\begin{aligned}
\mathbb{E}_v \left\{ J_\lambda(A + vv^T;\alpha) \right\} &= j_\lambda \sum_\mu u_{\mu,\lambda} \prod_i \frac{J_{(\mu_i)}(I;\alpha)\mathbb{E}_v\left\{ \widehat{J}_{(\mu_i)}(A + vv^T;\alpha)\right\}}{\alpha^{\mu_i}\mu_i!} \\
&= j_\lambda \sum_\mu u_{\mu,\lambda} \prod_i \frac{J_{(\mu_i)}(I;\alpha)}{\alpha^{\mu_i}\mu_i!} \sum_{k_i=0}^{\mu_i} \binom{\mu_i}{k_i} \widehat{J}_{(k_i)}(A;\alpha)\widehat{J}_{(\mu_i - k_i)}(ee^T;\alpha) \\
&= j_\lambda \sum_\mu u_{\mu,\lambda} \sum_{k_1=0}^{\mu_1} \cdots \sum_{k_n=0}^{\mu_n} c_{k_1,\ldots,k_n,\mu_1,\ldots,\mu_n} \prod_i J_{(k_i)}(A;\alpha) \\
&= j_\lambda \sum_\mu u_{\mu,\lambda} \sum_{k_1=0}^{\mu_1} \cdots \sum_{k_n=0}^{\mu_n} c_{k_1,\ldots,k_n,\mu_1,\ldots,\mu_n} \sum_\tau v_{\tau,\pi(\vec{k})} \frac{\alpha^{|\tau|}\prod_i \tau_i!}{j_{\pi(\vec{k})}} J_\tau(\vec{x};\alpha)
\end{aligned}$$

where $c_{k_1,\ldots,k_n,\mu_1,\ldots,\mu_n}$ is a nonzero constant and $\pi(\vec{k})$ denotes the partition formed that results from ordering the elements of the vector $\vec{k}$ in nonincreasing order.

By Theorem 2.4, we have that $u_{\mu,\lambda} \neq 0$ implies $\lambda \preceq \mu$, which, by Lemma 2.2, is equivalent to having $\mu' \preceq \lambda'$. Similar is true whenever $v_{\tau,\pi(\vec{k})} \neq 0$. Furthermore, the only partitions $\pi(k)$

and μ for which $c_{k_1,\dots,k_n,\mu_1,\dots,\mu_n} \neq 0$ is when the Young Tableaux of $\pi(\vec{k})$ is a sub-Tableaux of μ. Hence in particular, we must have

$$\pi(\vec{k})'_i \leq \mu'_i$$

for all i (which in turn implies weak majorization). Hence (14) implies

$$\tau' \preceq \pi(\vec{k})' \preceq_w \mu' \preceq \lambda'$$

which, by transitivity, implies the lemma. □

Note that, since $J_\tau(A;\alpha)$ is an orthogonal basis, (14) is equivalent to saying that the expansion of $\mathbb{E}\left\{J_\lambda(A + vv^T;\alpha)\right\}$ as a linear combination of the basis elements $J_\tau(A;\alpha)$ contains only terms which have $\tau' \preceq_w \lambda'$. As a sanity check, we note that when $\lambda = (k)$, we have $\lambda' = (1^k)$, and the only partitions that are weakly majorized by (1^k) are of the form (1^j) for $0 \leq j \leq k$. Hence the guarantee in Theorem 3.14 agrees with (13).

4 Open problems

We remark that the formula in (14) seems to be a property of the μ_n^α distributions rather than some superset of them. We conjecture the following:

Conjecture 4.1. *For all α, the distribution $\mu_n^{2/\alpha}$ is the only distribution which has equal marginals on its coordinates that satisfies (14).*

In the Appendix, we show some support for this conjecture — namely, that $\mu_n^{2/\alpha'}$ satisfies (14) if and only if $\alpha' = \alpha$. This is in sharp contrast to a number of other properties of μ_n^α that are invariant under changes in α (for example, the characteristic polynomial of $A + vv^T$ [12]). Certainly, an exact formula for (14) in the general case would also be interesting.

The other obvious open problems that is left by our work is one we find very interesting: finding constructions for producing other "uniformly distributed" objects in the general β case. Of particular interest (at least to the authors) is in the ability to satisfy Corollary 3.7. For example,

Problem 4.2. *Find a joint probability on pairs of unit vectors (u_1, u_2) such that*

- *u_1 and u_2 are orthogonal almost surely*
- *for all diagonal matrices A, the (random) matrix $B = su_1u_1^T + tu_2u_2^T$ satisfies*

$$\mathbb{E}_B\left\{\widehat{J}_\lambda(AB;\alpha)\right\} = \widehat{J}_\lambda(A;\alpha)\widehat{J}_\lambda(B;\alpha)$$

  for all partitions λ (and all $\alpha > 0$).

Comments in [8] suggest that such a distribution would likely need to depend on s and t as well as (possibly) A.

We are not so bold as to declare the distribution μ_n^α to mimic "the true" uniform distribution on the $\beta = 2/\alpha$ unit sphere, despite the compelling evidence provided by Corollary 3.7. While it is known that the collection of identities provided by Lemma 3.6 characterizes the Haar measure for $\beta = \{1, 2, 4\}$, Corollary 3.7 is only able to assert such an identity for the case when one of the matrices has rank 1. As this is our primary justification of treating these distributions as candidates for such a role, it would be interesting to know whether any other distributions satisfied the same identity (thereby providing another reasonable candidate), or whether the identities established by Corollary 3.7 were sufficient for characterizing the Haar measure for general β.

5 Acknowledgements

The authors would like to thank Roger Van Peski for his contributions in the early stages of this research. The calculations in the Appendix were performed using Mathematica.

6 Appendix

In this appendix, we show that the conclusion of Theorem 3.14 seems to be a characteristic of the distributions $\mu_n^{2/\alpha}$ and not some superset of them. For what follows, let μ be any distribution on unit vectors of length 2 for which

$$\mathbb{E}_v \left\{ v_1^2 \right\} = \mathbb{E}_v \left\{ v_2^2 \right\}. \tag{15}$$

Note that simply having equal marginals on the coordinates satisfies the condition. We also fix $\alpha = 1$.

Lemma 6.1. *If A is a 2×2 diagonal matrix and $v \sim \mu$, then*

$$\mathbb{E}_v \left\{ \left\langle \widehat{J}_4(A + vv^T; 1), \widehat{J}_{1^2}(A; 1) \right\rangle_1 = \frac{3}{5} \mathbb{E}_v \left\{ |v_1|^2 |v_2|^2 \right\} \right\} - \frac{1}{10}$$

Proof. Let

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}.$$

Then one can calculate that

$$\widehat{J}_4(A + vv^T; 1) = p(a, b) + q(a, b) \cos(2t) + r(a, b) \cos(4t)$$

where

$$p(a, b) = \widehat{J}_4(A; 1) + 2\widehat{J}_3(A; 1) + 2\widehat{J}_2(A; 1) + \widehat{J}_1(A; 1) + \frac{1}{5} + \frac{1}{120}(a - b)^2$$

and

$$r(a,b) = \frac{1}{40}(a-b)^2.$$

We now note that the property of μ having equal marginals implies

$$\int_0^{2\pi} v_1^2 \, d\mu(v) = \int_0^{2\pi} v_2^2 \, d\mu(v) = 0$$

and so $\cos(2t)$ will become 0 in expectation. So by the orthogonality property in Theorem 2.4, we get

$$\begin{aligned}
\mathbb{E}_v \left\{ \left\langle \widehat{J}_4(A+vv^T;1), \widehat{J}_{1^2}(A;1) \right\rangle_1 \right\} &= \mathbb{E}_v \left\{ \left\langle p(a,b) + r(a,b)\cos(4t), \widehat{J}_{1^2}(A;1) \right\rangle_1 \right\} \\
&= \mathbb{E}_v \left\{ \frac{1+3\cos(4t)}{120} \right\} \left\langle (a-b)^2, \widehat{J}_{1^2}(A;1) \right\rangle_1
\end{aligned}$$

and using the fact that $\cos(4t) = 1 - 8\cos(t)^2 \sin(t)^2$, we get that

$$1 - 3\cos(4t) = 4 - 24\cos(t)^2 \sin(t)^2 = 4(1 - 6|v_1|^2|v_2|^2).$$

Now it is easy to calculate that

$$(a-b)^2 = 3\widehat{J}_2(A;1) - 3\widehat{J}_{1^2}(A;1)$$

and so the claim follows by another application of Theorem 2.4. $\square$

Lemma 6.1 places a significant constraint on any μ which one might hope satisfies (14), and this is only one such constraint. In particular, in the case that $\mu = \mu_2^{2/\alpha'}$, we get by Corollary 3.3 and

$$\mathbb{E}_v \left\{ |v_1|^2 |v_2|^2 \right\} = \frac{B(\alpha'+1, \alpha'+1)}{B(\alpha', \alpha')} \overset{(4)}{=} \frac{\Gamma(\alpha'+1)^2}{\Gamma(2\alpha'+2)} \frac{\Gamma(2\alpha')}{\Gamma(\alpha')^2} = \frac{\alpha'}{4\alpha'+2}.$$

That is, $\alpha' = \alpha = 1$ is the *only* value of α' for which (14) can hold. In particular, when $\alpha = 2$, the distribution $\mu_n^{2/\alpha}$ is the Haar measure on the unit sphere of $\mathbb{R}^n$, which one would expect would exhibit any properties that were due to symmetries.

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